

# Gliwice PhD students of Professor Zahorski and scientific backgrounds of their thesis

Edyta Hetmaniok, Mariusz Pleszczyński and Roman Wituła

Professor Zahorski promoted in Gliwice two doctors. These are, in turn:

- Jerzy Timmler, Gliwice 1980,  
Uzbieżnienie szeregów wektorowych w przestrzeniach  $\mathbb{R}^n$  mnożnikami  $+1$  lub  $-1$   
(Generating the convergent vector series in spaces  $\mathbb{R}^n$  with multipliers  $+1$  or  $-1$ ).  
Reviewers: B. Jasek, T. Świątkowski, A. Wakulicz.
- Lucjan Meres, Gliwice 1979,  
O punktach osobliwych Pringsheima – Du Bois Reymonda funkcji dwu zmiennych  
(On singular points in the sense of Pringsheim – Du Bois Reymond of the functions  
of two variables).  
Reviewers: J. Lipiński, T. Świątkowski.

Let us discuss now briefly the contents of these thesis by extending slightly the objects of considerations with some additional facts and pieces of information.

## 1. Jerzy Timmler's PhD dissertation

For looking over the contents of J. Timmler's PhD dissertation few essential concepts need to be introduced, inventor of which was Ernst Steinitz<sup>1</sup> (1913, see [9, 19]).

**Definition 1.1.** *A unit vector  $v \in \mathbb{R}^n$ , bound at the origin, will be called a direction of convergence of series  $\sum u_k$  of vectors from  $\mathbb{R}^n$  if the series  $\sum |v \circ u_k|$  is convergent, where  $\circ$  denotes the inner product in  $\mathbb{R}^n$ .*

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<sup>1</sup> Ernst Steinitz, born in 1871 in Siemianowice Śląskie – Siemianowitz at this time (his father Sigmund Steinitz was born and worked in Gliwice (Gleiwitz at this time) which is of great symbolical importance for us – the authors). Great part of his scientific path is connected with Wrocław (Breslau at this time).

E. Hetmaniok, M. Pleszczyński, R. Wituła  
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,  
e-mail: {edyta.hetmaniok, mariusz.pleszczynski, roman.witula}@polsl.pl

R. Wituła, D. Słota, W. Hołubowski (eds.), *Monograph on the Occasion of 100<sup>th</sup> Birthday Anniversary of Zygmunt Zahorski*. Wydawnictwo Politechniki Śląskiej, Gliwice 2015, pp. 57–62.

**Definition 1.2.** If a unit vector  $v \in \mathbb{R}^n$ , bound at the origin, is not a direction of convergence of the given series  $\sum u_k$ , then it is called a direction of divergence of this series.

**Definition 1.3.** A unit vector  $v \in \mathbb{R}^n$ , bound at the origin, is called a principal direction of series  $\sum u_k$  if each open circular cone with vertex at the origin, containing vector  $v$ , is such that the sum of absolute values of terms  $u_k$ , belonging to this region, is equal to  $\infty$ .

**Definition 1.4.** Let  $\mathbf{u} = \{u_k\}$  be a sequence of vectors from  $\mathbb{R}^n$  such that

$$\lim u_k = \mathbb{O} \quad \text{and} \quad \sum \|u_k\| = \infty.$$

By  $S(\mathbf{u})$  we denote the set of sums of all convergent series  $\sum_{k=1}^{\infty} m_k u_k$ , where  $m_k \in \{-1, 1\}$  for every  $k \in \mathbb{N}$ .

J. Timmler's PhD dissertation is mostly devoted to the discussion of sets  $S(\mathbf{u})$  in case of  $n \geq 2$ . Let us recall that from the Riemann Derangement Theorem it follows that  $S(\mathbf{u}) = \mathbb{R}$  always in case of  $n = 1$ . Dvoretzky and Hanani in [4] proved that  $S(\mathbf{u})$  is nonempty in case of  $n = 2$ . Timmler generalized this result in his PhD dissertation for the cases of any  $n \in \mathbb{N}$ .

Hanani in paper [7] proved, in case of  $n = 2$ , that each sequence  $\mathbf{u}$  having at least one pair of linearly independent principal directions possesses  $S(\mathbf{u}) = \mathbb{R}^2$ . Furthermore, Timmler generalized this result in his PhD dissertation for the case of  $n = 3$ . More precisely, he proved by using the barycentric coordinates method that if the sequence  $\mathbf{u}$  has at least one triple of linearly independent principal directions, then  $S(\mathbf{u}) = \mathbb{R}^3$ .

Let us notice here that the simple adaptation of the proof of Theorem 2 from Jasek's paper [9] enables to obtain the following results.

**Theorem 1.5.** If series  $\sum u_k$  of vectors from  $\mathbb{R}^n$  is not absolutely convergent, then it has at most  $n$  directions of convergence. Moreover, the set of principal directions of this series is closed.

**Remark 1.6.** Good supplement for the presented above subject matter can be given by Bronisław Jasek's<sup>2</sup> papers [10, 11, 9]. Since majority of these results is located in the field of complex series, therefore we felt an irresistible desire to generalize these results for the series in  $\mathbb{R}^n$ . It does not seem to be so easy, however it is very tempting for sure. Additionally we note that A. Sowa obtained in [18] some interesting result on the property of set  $S(\mathbf{u})$  for the series  $\sum u_k$  in  $\mathbb{R}^3$  possessing two principal directions and a direction of convergence all linearly independent in  $\mathbb{R}^3$ .

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<sup>2</sup> Bronisław Jasek (born in 1930). He defended his PhD thesis in 1962 at the Wrocław University (supervised by Edward Marczewski). Docent in the Institute of Mathematics at the Wrocław University of Technology since 1968. He hold the position of Dean in the Faculty of Fundamental Problems of Technology at the Wrocław University of Technology (1968–1975), and then the position of Director of the Institute of Mathematics at the Wrocław University of Technology. He is retired now for several years. He lives in Wrocław.

In this moment we would like to express our thanks to Professor Zbigniew Skoczylas for the information about Professor Bronisław Jasek.

In J. Timmler's PhD dissertation the following theorem is proven as well.

**Theorem 1.7.** *For every  $n \in \mathbb{N}$  there exists a positive real number  $C(n)$  such that for every finite sequence  $\{v_k\}_{k=1}^N$  of vectors from  $\mathbb{R}^n$ , with lengths  $\leq 1$ , the multipliers  $\varepsilon_k \in \{-1, 1\}$ ,  $k = 1, 2, \dots, N$  exist such that*

$$\left\| \sum_{k=1}^r \varepsilon_k v_k \right\| \leq C(n),$$

for every  $r = 1, \dots, N$ .

**Corollary 1.8.** *For every sequence  $\{v_k\}_{k=1}^\infty$  of vectors from  $\mathbb{R}^n$ , with lengths  $\|v_k\| \rightarrow 0$ , there exists a sequence  $\{\varepsilon_k\}_{k=1}^\infty$  of multipliers, where  $\varepsilon_k \in \{-1, 1\}$  for every  $k = 1, 2, \dots$ , such that the series  $\sum \varepsilon_k v_k$  is convergent in  $\mathbb{R}^n$ .*

**Remark 1.9.** The above Theorem and Corollary have been proven independently by many authors (see, for example, Wituła problem 1.2.14 in [6], Calabi and Dvoretzky [2], J. Timmler proved this theorem in [21] only in case of  $n = 3$  but by applying the very interesting geometric considerations).

Moreover, we know that  $C(1) \leq 1$ ,  $C(2) \leq \sqrt{3}$  (see [4], simultaneously let us notice that in paper [4] there were presented only the ideas of proof of this inequality, whereas the full proof was published by J. Timmler in paper [20]) and  $C(3) \leq 12$  (see [21]).

Corollary 1.8 holds also in space  $\mathbb{R}^\infty$  (Y. Katznelson and O.C. McGehee [12]), however it is not true in the case of every infinitely dimensional Banach space (see [3] p. 157, Theorem 8).

**Remark 1.10.** In reference to Corollary 1.8 let us recall the following concept (see [2]).

A set  $S$  of complex numbers is called a sum factor set if for any sequence  $\{a_n\}_{n=1}^\infty$  of complex numbers such that

$$\lim a_n = 0$$

and

$$\sum |a_n| = \infty,$$

and for any complex number  $\mathbf{s}$  there exists a sequence  $\{\xi_n\}_{n=1}^\infty$ ,  $\xi_n \in S$ ,  $n \in \mathbb{N}$ , for which  $\sum_{n=1}^\infty \xi_n a_n = \mathbf{s}$ .

H. Hornich proved in [8] that for every  $k \geq 3$  the set  $S_k$  of  $k$ -th roots of unity is the sum factor set. It is obvious that the set  $\{-1, 1\}$  is not the sum factor set. We know [2] that a bounded set  $M \subset \mathbb{C}$  is the sum factor set if and only if  $\mathbb{O} \in \text{int}(\text{conv } M)$ .

**Remark 1.11.** There exists some subtle connection between Theorem 1.7 and Corollary 1.8 and the respective theorem and corollary concerning the arrangements of finite and infinite series. We present them now, but by omitting the proof of technical nature of the following theorem.

**Theorem 1.12.** *For every  $n \in \mathbb{N}$  and finite sequence  $\{u_k\}_{k=1}^N$  of vectors from  $\mathbb{R}^n$ , such that  $\sum_{k=1}^N u_k = \mathbb{O}$ , there exists a permutation  $p$  of set  $\{1, 2, \dots, N\}$  such that*

$$\left\| \sum_{k=1}^m u_{p(k)} \right\| \leq \sqrt{\sum_{k=1}^N \|u_k\|^2} \quad (1)$$

for every  $m = 1, 2, \dots, N$ .

**Corollary 1.13.** For every  $n \in \mathbb{N}$  and for every sequence  $\{u_k\}_{k=1}^\infty$  of vectors from  $\mathbb{R}^n$ , such that

$$\{u_k\}_{k=1}^\infty \in l_{\mathbb{R}}^2(\mathbb{R}^n) \quad (2)$$

and

$$\sum_{k=1}^N u_k = \mathbb{O}$$

for infinitely many indices  $N \in \mathbb{N}$ , there exists a permutation  $p$  of  $\mathbb{N}$  such that

$$\sum_{k=1}^\infty u_{p(k)} = \mathbb{O}.$$

*Proof.* (of Corollary 1.13)

First we choose by induction an increasing sequence  $\{N_i\}_{i=1}^\infty$  of positive integers satisfying the following conditions

$$\sum_{k=1}^{N_i} u_k = \mathbb{O}, \quad (3)$$

$$\sum_{k=N_i}^\infty \|u_k\|^2 \leq 4^i,$$

for every  $i \in \mathbb{N}$ . By Theorem 1.12, for every  $i \in \mathbb{N}$  there exists a permutation  $p_i$  of set  $\{1 + N_i, 2 + N_i, \dots, N_{i+1}\}$  such that

$$\left\| \sum_{k=1+N_i}^m u_{p_i(k)} \right\| \leq \sqrt{\sum_{k=1+N_i}^{N_{i+1}} \|u_k\|^2} \leq 2^i, \quad (4)$$

for every  $m = 1 + N_i, 2 + N_i, \dots, N_{i+1}$ .

The desired permutation  $p$  of  $\mathbb{N}$  could be defined by formula

$$p(k) = \begin{cases} k & \text{for every } k = 1, 2, \dots, N_1, \\ p_i(k) & \text{for every } k = 1 + N_i, 2 + N_i, \dots, N_{i+1}, \\ & \text{and } i \in \mathbb{N}. \end{cases}$$

Immediately from (3) and (4) it follows that  $\sum_{k=1}^\infty u_{p(k)} = \mathbb{O}$ . □

**Remark 1.14.** In the course of elaborating Theorem 1.12 and Corollary 1.13 a discussion between the authors started about the possibility of substituting condition (2) (resulting directly from inequality (1) of Theorem 1.12) with the weaker condition of the form  $\lim_{k \rightarrow \infty} u_k = \mathbb{O}$ . We had no any suitable proof, however completely accidentally

in paper [5] the author – Wilhelm Gross – proved (by induction but with application of some very nice geometrical considerations) the following fact (which enables to substitute condition (2) of Corollary 1.13 with condition  $\lim_{k \rightarrow \infty} u_k = \mathbb{O}$ ):

**Theorem 1.15.** *For every  $n \in \mathbb{N}$  and finite sequence  $\{u_k\}_{k=1}^N$  of vectors from  $\mathbb{R}^n$  such that  $\sum_{k=1}^N u_k = \mathbb{O}$ , there exists a permutation  $q$  of set  $\{1, 2, \dots, N\}$  such that*

$$\left\| \sum_{k=1}^m u_{q(k)} \right\| \leq (2^n - 1) \max\{\|u_k\| : k = 1, 2, \dots, N\},$$

for every  $m = 1, 2, \dots, N$ .

**Remark 1.16.** In paper [12] (see Lemma 4) it is proven that for the series discussed in Corollary 1.13 the set of sums of convergent series of the form  $\sum_{k=1}^{\infty} u_{p(k)}$ , where  $p$  is a permutation of  $\mathbb{N}$ , is the linear subspace of  $\mathbb{R}^n$ .

## 2. Lucjan Meres' PhD dissertation

The second PhD student of Professor Zahorski, Lucjan Meres, found in his PhD dissertation the complete characterization of the set of singular points of the functions of two variables not possessing the Cauchy singularities (i.e. points singular in the sense of Cauchy). Later, in paper [15], Meres generalized this characterization for the functions of several variables (not possessing the Cauchy singularities). Results from the Meres' PhD dissertation have been also partly published in papers [13, 14].

Generalization of the Meres theorems and the appropriate Zahorski theorem (which, in fact, constituted the groundwork of Meres' research) has been given by Józef Siciak in paper [17] (professor W. Wilczyński referred to this one as well in his paper included in this monograph). J. Schmets and M. Valdivia in paper [16] proved, in order to extend Siciak's result, that for every  $\gamma \in (0, \infty)$  and  $n \in \mathbb{N}$  the Zahorski theorem has a solution  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  belonging to the Gevrey class  $\Gamma_\gamma$ , that is to the set of  $f \in C_\infty(\mathbb{R}^n)$  for which there exists  $a, b > 0$  such that

$$\|D^\alpha f\|_{\mathbb{R}^n} \leq a b^{|\alpha|} (|\alpha|!)^\gamma$$

for every  $\alpha \in \mathbb{N}_0^n$ .

**Remark.** All cited here papers published in the Scientific Notes of Silesian University of Technology (Zeszyty Naukowe Politechniki Śląskiej in Polish) will be very soon scanned and available online.

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